

ŁOJASIEWICZ IDEALS IN DENJOY-CARLEMAN CLASSES

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ABSTRACT. The classical notion of Łojasiewicz ideals of smooth functions is studied in the context of non-quasianalytic Denjoy-Carleman classes. In the case of principal ideals, we obtain a characterization of Łojasiewicz ideals in terms of properties of a generator. This characterization involves a certain type of estimates that differ from the usual Łojasiewicz inequality. We then show that basic properties of Łojasiewicz ideals in the \mathcal{C}^∞ case have a Denjoy-Carleman counterpart.

INTRODUCTION

Let Ω be an open subset of \mathbb{R}^n , and let $\mathcal{C}^\infty(\Omega)$ be the Fréchet algebra of smooth functions in Ω . Let X be a closed subset of Ω . An element φ of $\mathcal{C}^\infty(\Omega)$ is said to satisfy the *Łojasiewicz inequality with respect to X* if, for every compact subset K of Ω , there are real constants $C > 0$ and $\nu \geq 1$ such that, for any $x \in K$, we have

$$(1) \quad |\varphi(x)| \geq C \operatorname{dist}(x, X)^\nu.$$

An element of $\mathcal{C}^\infty(\Omega)$ is said to be *flat on X* if it vanishes, together with all its derivatives, at each point of X . Denote by \underline{m}_X^∞ the ideal of functions of $\mathcal{C}^\infty(\Omega)$ that are flat on X . The following statement appears in Section V.4 of [15] and establishes a connection between the Łojasiewicz inequality for functions and the behavior of ideals with respect to flat functions.

Theorem 0.1. *Let \mathcal{I} be a finitely generated proper ideal in $\mathcal{C}^\infty(\Omega)$, and let X be the zero set of \mathcal{I} . The following properties are equivalent:*

- (A) *The ideal \mathcal{I} contains an element φ which satisfies the Łojasiewicz inequality with respect to X ,*
- (B) *One has $\underline{m}_X^\infty \subset \mathcal{I}$,*
- (C) *One has $\underline{m}_X^\infty = \mathcal{I}\underline{m}_X^\infty$.*

A finitely generated ideal \mathcal{I} satisfying the equivalent properties (A), (B), (C) is called a *Łojasiewicz ideal*. A principal ideal is Łojasiewicz if and only if condition (A) holds for a generator φ of the ideal. In the general case of a finitely generated ideal with generators $\varphi_1, \dots, \varphi_p$, one can take $\varphi = \varphi_1^2 + \dots + \varphi_p^2$. Łojasiewicz ideals play an important role in the study of ideals of differentiable functions, see for instance [8, 14, 15]. In particular, every

closed ideal of finite type is Łojasiewicz, whereas the converse statement is false.

In the present paper, we study a possible approach to Łojasiewicz ideals in non-quasianalytic Denjoy-Carleman classes $\mathcal{C}_M(\Omega)$. While several papers have already been devoted to the study of closed ideals in $\mathcal{C}_M(\Omega)$ (see for example [10, 11, 12]), a suitable notion of Łojasiewicz ideal is still lacking, even in the case of principal ideals. This is due to the fact that if we put $\underline{m}_{X,M}^\infty = \underline{m}_X^\infty \cap \mathcal{C}_M(\Omega)$ and $\mathcal{I} = \varphi \mathcal{C}_M(\Omega)$, where φ is a given element of $\mathcal{C}_M(\Omega)$, it turns out that the usual Łojasiewicz inequality (1) is not a sufficient condition for the inclusion $\mathcal{I} \subset \underline{m}_{X,M}^\infty$, let alone for the equality $\underline{m}_{X,M}^\infty = \mathcal{I} \underline{m}_{X,M}^\infty$. Therefore, it is natural to ask for a characterization of both of these properties in terms of the generator φ , in the spirit of the characterization given by Theorem 0.1 in the \mathcal{C}^∞ case.

In the case of principal ideals, a suitable characterization will be obtained in Theorem 2.4. In the statement, the Łojasiewicz inequality (1) has to be replaced by a quite different property involving successive derivatives of $1/\varphi$, which will be shown to be equivalent to the obvious Denjoy-Carleman version of property (C), that is, to the equality $\underline{m}_{X,M}^\infty = \mathcal{I} \underline{m}_{X,M}^\infty$. We are also able to get an equivalence with a corresponding version of property (B), provided we consider the inclusion $\underline{m}_{X,M}^\infty \subset \mathcal{I}$ together with a mild extra requirement on the flat points of φ .

In order to prove these results, one has to deal with the fact that the constructive techniques used by Tougeron in the classical \mathcal{C}^∞ case don't seem applicable to the \mathcal{C}_M setting. Thus, the main part of our proof of Theorem 2.4 is actually based on a functional-analytic argument. Once the theorem is proven, we discuss several related properties showing that basic results of the \mathcal{C}^∞ case are extended in a consistent way. For instance, we show that our \mathcal{C}_M Łojasiewicz condition holds for closed principal ideals, and we also provide a non-closed example.

1. DENJOY-CARLEMAN CLASSES

1.1. Notation. For any multi-index $J = (j_1, \dots, j_n)$ of \mathbb{N}^n , we always denote the length $j_1 + \dots + j_n$ of J by the corresponding lower case letter j . We put $J! = j_1! \dots j_n!$, $D^J = \partial^j / \partial x_1^{j_1} \dots \partial x_n^{j_n}$ and $x^J = x_1^{j_1} \dots x_n^{j_n}$. We denote by $|\cdot|$ the euclidean norm on \mathbb{R}^n ; balls and distances in \mathbb{R}^n will always be considered with respect to that norm.

If a is a point in \mathbb{R}^n , and if f is a smooth function in a neighborhood of a , we denote by $T_a f$ the formal Taylor series of f at a , that is, the element of $\mathbb{C}[[x_1, \dots, x_n]]$ defined by

$$T_a f = \sum_{J \in \mathbb{N}^n} \frac{1}{J!} D^J f(a) x^J.$$

The function f is said to be *flat* at the point a if $T_a f = 0$.

1.2. Some properties of sequences. Let $M = (M_j)_{j \geq 0}$ be a sequence of real numbers satisfying the following assumptions:

(2) the sequence M is increasing, with $M_0 = 1$,

(3) the sequence M is *logarithmically convex*.

Property (3) amounts to saying that M_{j+1}/M_j is increasing. Together with (2), it implies

$$(4) \quad M_j M_k \leq M_{j+k} \text{ for any } (j, k) \in \mathbb{N}^2.$$

We say that the *moderate growth* property holds if there is a constant $A > 0$ such that, conversely,

$$(5) \quad M_{j+k} \leq A^{j+k} M_j M_k \text{ for any } (j, k) \in \mathbb{N}^2.$$

We say that M satisfies the *strong non-quasianalyticity* condition if there is a constant $A > 0$ such that

$$(6) \quad \sum_{j \geq k} \frac{M_j}{(j+1)M_{j+1}} \leq A \frac{M_k}{M_{k+1}} \text{ for any } k \in \mathbb{N}.$$

Notice that property (6) is indeed stronger than the classical Denjoy-Carleman quasianalyticity condition

$$(7) \quad \sum_{j \geq 0} \frac{M_j}{(j+1)M_{j+1}} < \infty.$$

The sequence M is said to be *strongly regular* if it satisfies (2), (3), (5) and (6).

Example 1.3. Let α and β be real numbers, with $\alpha > 0$. The sequence M defined by $M_j = j!^\alpha (\ln(j+e))^{\beta j}$ is strongly regular. This is the case, in particular, for Gevrey sequences $M_j = j!^\alpha$.

With every sequence M satisfying (2) and (3) we also associate the function h_M defined by $h_M(t) = \inf_{j \geq 0} t^j M_j$ for any real $t > 0$, and $h_M(0) = 0$. From (2) and (3), it is easy to see that the function h_M is continuous, increasing, and it satisfies $h_M(t) > 0$ for $t > 0$ and $h_M(t) = 1$ for $t \geq 1/M_1$. It also fully determines the sequence M , since we have $M_j = \sup_{t > 0} t^{-j} h_M(t)$.

Example 1.4. Let M be as in Example 1.3, and put $\eta(t) = \exp(-(t|\ln t|^\beta)^{-1/\alpha})$ for $t > 0$. Elementary computations show that there are constants $a > 0$, $b > 0$ such that $\eta(at) \leq h_M(t) \leq \eta(bt)$ for $t \rightarrow 0$.

A technically important consequence of the moderate growth assumption (5) is the existence of a constant $\rho \geq 1$, depending only on M , such that

$$(8) \quad h_M(t) \leq (h_M(\rho t))^2 \text{ for any } t \geq 0.$$

We refer to [3] for a proof of the implication (5) \Rightarrow (8).

1.5. Denjoy-Carleman classes. Let Ω be an open subset of \mathbb{R}^n , and let M be a sequence of real numbers satisfying (2) and (3). We define $\mathcal{C}_M(\Omega)$ as the space of functions f belonging to $\mathcal{C}^\infty(\Omega)$ and such that, for any compact subset K of Ω , one can find a real $\sigma > 0$ and a constant $C > 0$, such that the estimate

$$(9) \quad |D^J f(x)| \leq C \sigma^j j! M_j$$

holds for any multi-index $J \in \mathbb{N}^n$ and any $x \in K$.

Given a function f in $\mathcal{C}^\infty(\Omega)$, a compact subset K of Ω and a real number $\sigma > 0$, put

$$\|f\|_{K,\sigma} = \sup_{x \in K, J \in \mathbb{N}^n} \frac{|D^J f(x)|}{\sigma^j j! M_j}.$$

We see that f belongs to $\mathcal{C}_M(\Omega)$ if and only if, for any compact subset K of Ω , one can find a real $\sigma > 0$ such that $\|f\|_{K,\sigma}$ is finite ($\|f\|_{K,\sigma}$ then coincides with the smallest constant C for which (9) holds). The function space $\mathcal{C}_M(\Omega)$ is called the *Denjoy-Carleman class of functions of class \mathcal{C}_M in the sense of Roumieu* (which corresponds to $\mathcal{E}_{\{j!M_j\}}(\Omega)$ in the notation of [5]).

From now on, we will assume that the sequence M is strongly regular. In particular, it satisfies (7), which implies that $\mathcal{C}_M(\Omega)$ contains compactly supported functions. We denote by $\mathcal{D}_M(\Omega)$ the space of elements of $\mathcal{C}_M(\Omega)$ with compact support in Ω .

For the reader's convenience, we now recall some basic topological facts about $\mathcal{C}_M(\Omega)$ and $\mathcal{D}_M(\Omega)$, without proof (we refer to [5] for the details). With each Whitney 1-regular compact subset K of Ω , and each integer $\nu \geq 1$, we associate the vector space $\mathcal{C}_{M,K,\nu}$ of all functions f which are \mathcal{C}^∞ -smooth on K in the sense of Whitney, and such that $\|f\|_{K,\nu} < \infty$. Then $\mathcal{C}_{M,K,\nu}$ is a Banach space for the norm $\|\cdot\|_{K,\nu}$ and it can be shown that for $\nu < \nu'$, the inclusion $\mathcal{C}_{M,K',\nu} \hookrightarrow \mathcal{C}_{M,K',\nu'}$ is compact. We define the Denjoy-Carleman class $\mathcal{C}_M(K)$ as the reunion of all spaces $\mathcal{C}_{M,K,\nu}$ with $\nu \geq 1$. Endowed with the inductive topology, $\mathcal{C}_M(K)$ is a (DFS)-space (or *Silva space*). Given an exhaustion $(K_j)_{j \geq 1}$ of Ω by Whitney 1-regular compact subsets, the Denjoy-Carleman class $\mathcal{C}_M(\Omega)$ can be identified with the projective limit of all (DFS)-spaces $\mathcal{C}_M(K_j)$.

Similarly, denote by $\mathcal{D}_{M,K,\nu}$ the space of all functions $f \in \mathcal{C}^\infty(\Omega)$ such that $\text{supp } f \subset K$ and $\|f\|_{K,\nu} < \infty$. Then $\mathcal{D}_{M,K,\nu}$ is a Banach space and we have the following properties: for $K \subset K'$, the space $\mathcal{D}_{M,K,\nu}$ is a closed subspace of $\mathcal{D}_{M,K',\nu}$, and for $\nu < \nu'$, the inclusion $\mathcal{D}_{M,K',\nu} \hookrightarrow \mathcal{D}_{M,K',\nu'}$ is compact. For any integer $\nu \geq 1$, put $\mathcal{D}_\nu = \mathcal{D}_{M,K_\nu,\nu}$, $\|\cdot\|_\nu = \|\cdot\|_{K_\nu,\nu}$, and notice that we have $\mathcal{D}_M(\Omega) = \bigcup_{\nu \geq 1} \mathcal{D}_\nu$ as a set. By the preceding remarks, we have a compact injection $\mathcal{D}_\nu \hookrightarrow \mathcal{D}_{\nu+1}$. Thus, the space $\mathcal{D}_M(\Omega)$ is another (DFS)-space for the corresponding inductive limit topology.

1.6. Some basic properties of $\mathcal{C}_M(\Omega)$. Properties (2) and (3) of the sequence M ensure that $\mathcal{C}_M(\Omega)$ is an algebra containing the algebra of real-analytic functions, and that \mathcal{C}_M regularity is stable under composition [9]. The latter result implies, in particular, the following invertibility property.

Lemma 1.7 ([9]). *If the function f belongs to $\mathcal{C}_M(\Omega)$ and has no zero in Ω , then the function $1/f$ belongs to $\mathcal{C}_M(\Omega)$.*

It is also known that the implicit function theorem holds within the framework of \mathcal{C}_M regularity [6]. Thus, \mathcal{C}_M manifolds and submanifolds can be defined in the usual way.

The strong regularity assumption on M ensures that suitable versions of Whitney's extension theorem and Whitney's spectral theorem hold in $\mathcal{C}_M(\Omega)$; see [1, 2, 3, 4]. The extension result relies on a crucial construction of cutoff functions whose successive derivatives satisfy a certain type of optimal estimates. This construction is due to Bruna [2], see also Proposition 4 of [3]. Up to a rescaling in the statement of [3], the result can be written as follows.

Lemma 1.8 ([2, 3]). *There is a constant $c > 0$ such that, for any real numbers $r > 0$ and $\sigma > 0$, one can find a function $\chi_{r,\sigma}$ belonging to $\mathcal{C}_M(\mathbb{R}^n)$, compactly supported in the ball $B = B(0, r)$, and such that we have $0 \leq \chi_{r,\sigma} \leq 1$, $\chi_{r,\sigma}(t) = 1$ for $|t| \leq r/2$ and $\|\chi_{r,\sigma}\|_{\overline{B}, c\sigma} \leq (h_M(\sigma r))^{-1}$.*

We shall also need a basic result on flat functions. Given a closed subset Z of Ω , recall that $\underline{m}_{Z,M}^\infty$ denotes the ideal of functions of $\mathcal{C}_M(\Omega)$ which are flat at each point of Z .

Lemma 1.9. *Let f be an element of $\underline{m}_{Z,M}^\infty$. For any compact subset K of Ω , there are positive constants c_1 and c_2 such that, for any multi-index I in \mathbb{N}^n and any x in K , we have*

$$(10) \quad |D^I f(x)| \leq c_1 c_2^i i! M_i h_M(c_2 \text{dist}(x, Z)).$$

Proof. For any real $r > 0$, put $K_r = \{y \in \Omega : \text{dist}(y, K) \leq r\}$. If r is chosen small enough, K_r is a compact subset of Ω . Thus, there is a constant $\sigma > 0$ such that, for any $y \in K_r$, $I \in \mathbb{N}^n$ and $J \in \mathbb{N}^n$, we have $|D^{I+J} f(y)| \leq \|f\|_{K_r, \sigma} \sigma^{i+j} (i+j)! M_{i+j}$. Using (5) and the elementary estimate $(i+j)! \leq 2^{i+j} i! j!$, we get

$$(11) \quad |D^{I+J} f(y)| \leq c_1 c_2^i i! M_i c_2^j j! M_j.$$

with $c_1 = \|f\|_{K_r, \sigma}$ and $c_2 = 2A\sigma$. Now let x be a point in K , and let z be a point in Z such that

$$(12) \quad |x - z| = \text{dist}(x, Z).$$

If $\text{dist}(x, Z) \leq r$, then the segment $[x, z]$ is contained in K_r . Since $D^I f$ is flat at z , the Taylor formula then yields $|D^I f(x)| \leq \sup_{y \in K_r} |D^{I+J} f(y)| |x - z|^j / j!$ for any $J \in \mathbb{N}^n$. Using (11) and (12), and taking the infimum with respect to J , we obtain (10). If $\text{dist}(x, Z) > r$, the estimate is a simple consequence of the definition of $\mathcal{C}_M(\Omega)$, up to a modification of c_1 and c_2 . \square

2. ŁOJASIEWICZ IDEALS

The following notion will serve as a replacement for the standard Łojasiewicz inequality.

Definition 2.1. Let φ be a non-zero element of $\mathcal{C}_M(\Omega)$ and let X be the zero set of φ . We say that φ satisfies the \mathcal{C}_M Łojasiewicz condition if, for any compact subset K of Ω and any real $\lambda > 0$, one can find positive constants C and σ (depending on K and λ) such that, for any multi-index $J \in \mathbb{N}^n$ and any $x \in K \setminus X$, we have

$$(13) \quad \left| D^J(1/\varphi)(x) \right| \leq \frac{C \sigma^{|J|} M_J}{h_M(\lambda \operatorname{dist}(x, X))}.$$

Remark 2.2. From the basic properties of h_M in Section 1.2, we see that, on a given open subset $\{x \in \Omega : \operatorname{dist}(x, X) > \delta\}$ with $\delta > 0$, the \mathcal{C}_M Łojasiewicz condition amounts to nothing more than the conclusion of Lemma 1.7. It is only relevant as a bound on the explosion of $1/\varphi$ and its derivatives in a neighborhood of the zeros of φ .

In Section 3, we will provide examples of functions for which the \mathcal{C}_M Łojasiewicz condition holds. Lemma 2.3 below shows that such functions cannot have “too many flat points” on the boundary of their zero set.

Lemma 2.3. Let φ be a non-zero element of $\mathcal{C}_M(\Omega)$ and let X be its zero set. Assume that φ satisfies the \mathcal{C}_M Łojasiewicz condition, and let $X_\infty = \{a \in X : T_a \varphi = 0\}$ be the set of points of flatness of φ . Then $X \setminus X_\infty$ is dense in the boundary ∂X of X .

Proof. Notice that φ is necessarily flat at each interior point of X , hence the inclusion $X \setminus X_\infty \subset \partial X$. We prove the density property by contradiction. If the property is not true, there are a point a in ∂X and an open neighborhood ω of a in Ω , such that φ is flat on $\omega \cap \partial X$. Put $K = \overline{B(a, r)}$ with $r = \frac{1}{2} \operatorname{dist}(a, X \setminus \omega)$. Then K is a compact subset of ω and we have

$$(14) \quad \operatorname{dist}(x, \omega \cap \partial X) = \operatorname{dist}(x, \partial X) = \operatorname{dist}(x, X) \text{ for any } x \in K.$$

Using Lemma 1.9 on the open set ω , with $f = \varphi|_\omega$, $Z = \omega \cap \partial X$ and $I = 0$, we see that there are constants c_1 and c_2 such that we have $|\varphi(x)| \leq c_1 h_M(c_2 \operatorname{dist}(x, \omega \cap \partial X))$ for any $x \in K$. Taking property (8) into account, we obtain, for any $x \in K$,

$$(15) \quad |\varphi(x)| \leq c_1 h_M(c_3 \operatorname{dist}(x, \omega \cap \partial X))^2$$

with $c_3 = \rho c_2$. On the other hand, using the \mathcal{C}_M Łojasiewicz condition with $\lambda = c_3$ and $J = 0$, we obtain a constant $c_4 > 0$ such that, for any $x \in K \setminus X$,

$$(16) \quad |\varphi(x)| \geq c_4 h_M(c_3 \operatorname{dist}(x, X)).$$

Gathering (14), (15) and (16), we obtain $h_M(c_3 \operatorname{dist}(x, X)) \geq c_4/c_1$ for any $x \in K \setminus X$, which is impossible since $K \setminus X$ has at least an accumulation point on X , namely the point a . \square

We are now able to state the main result.

Theorem 2.4. Let φ be a non-zero element of $\mathcal{C}_M(\Omega)$, let X be its zero set, and let X_∞ be its set of points of flatness. Put $\mathcal{I} = \varphi \mathcal{C}_M(\Omega)$. The following

properties are equivalent:

(A') The function φ satisfies the \mathcal{C}_M Łojasiewicz condition,

(B') One has $\underline{m}_{X,M}^\infty \subset \mathcal{I}$ and $X \setminus X_\infty$ is dense in ∂X ,

(C') One has $\underline{m}_{X,M}^\infty = \mathcal{I}\underline{m}_{X,M}^\infty$.

Proof. We prove the implication (C') \Rightarrow (A') first. We use the (DFS)-space $\mathcal{D}_M(\Omega) = \varinjlim \mathcal{D}_\nu$ defined in Section 1.5. The intersection $\mathcal{D}_M(\Omega) \cap \underline{m}_{X,M}^\infty$ is obviously closed in $\mathcal{D}_M(\Omega)$, hence it is also a (DFS)-space with step spaces $\mathcal{E}_\nu = \mathcal{D}_\nu \cap \underline{m}_{X,M}^\infty$.

It is easy to see that the map $\Lambda : \mathcal{D}_M(\Omega) \cap \underline{m}_{X,M}^\infty \rightarrow \mathcal{D}_M(\Omega) \cap \underline{m}_{X,M}^\infty$ defined by $\Lambda(f) = \varphi f$ is continuous. Moreover, given an element g of $\mathcal{D}_M(\Omega) \cap \underline{m}_{X,M}^\infty$, the assumption implies that it can be written φh for some $h \in \underline{m}_{X,M}^\infty$. If χ is an element of $\mathcal{D}_M(\Omega)$ such that $\chi = 1$ on $\text{supp } g$, we then have $g = \chi g = \varphi f$ with $f = \chi h \in \mathcal{D}_M(\Omega) \cap \underline{m}_{X,M}^\infty$. Thus, Λ is also surjective.

We can therefore apply the De Wilde open mapping theorem ([7], Chapter 24), which yields the following property: for any $\nu \geq 1$, there exist an integer $\mu_\nu \geq 1$ and a real constant $C_\nu > 0$ such that, for any $g \in \mathcal{E}_\nu$, one can find an element f of \mathcal{E}_{μ_ν} such that

$$(17) \quad \varphi f = g \quad \text{and} \quad \|f\|_{\mu_\nu} \leq C_\nu \|g\|_\nu.$$

Now, let x be a point in $K \setminus X$, let d_K be a real number such that $0 < d_K < \text{dist}(K, \mathbb{R}^n \setminus \Omega)$, and put $r_x = \min(\text{dist}(x, X), d_K)$. Given $\lambda > 0$, we apply Lemma 1.8 with $r = 2r_x/3$ and $\sigma = 3\lambda/2$. We set $g_x(y) = \chi_{r,\sigma}(y - x)$. Then g_x belongs to $\mathcal{C}_M(\Omega)$ and is compactly supported in the ball $B_x = B(x, 2r_x/3)$. Obviously B_x is contained in $K' = \{y \in \Omega : \text{dist}(y, K) \leq 2d_K/3\}$, which is a compact subset of Ω . For a sufficiently large integer ν , depending only on K and λ , we have $\nu \geq c\sigma$ and $K' \subset K_\nu$, so that g_x belongs to \mathcal{E}_ν and

$$(18) \quad \|g_x\|_\nu = \|g_x\|_{\overline{B_x}, \nu} \leq \|g_x\|_{\overline{B_x}, c\sigma} \leq (h_M(\lambda r_x))^{-1}.$$

Since $h_M(\lambda r_x)$ equals either $h_M(\lambda \text{dist}(x, X))$ or $h_M(\lambda d_K)$, and since $h_M(t) \leq 1$ holds for every $t > 0$, we also have

$$(19) \quad h_M(\lambda r_x) \geq h_M(\lambda d_K) h_M(\lambda \text{dist}(x, X)).$$

Now, if f_x denotes the element of \mathcal{E}_{μ_ν} associated with g_x by property (17), we therefore have $\varphi f_x = g_x$ and, thanks to (18) and (19),

$$(20) \quad \|f_x\|_{\mu_\nu} \leq C'_\nu (h_M(\lambda \text{dist}(x, X)))^{-1}$$

with $C'_\nu = C_\nu/h_M(\lambda d_K)$. For any y in $B'_x = B(x, r_x/3)$, we have $g_x(y) = 1$, hence

$$(21) \quad f_x(y) = 1/\varphi(y).$$

In particular, we have $f_x(y) \neq 0$. Thus, we derive $B'_x \subset \text{supp } f_x \subset K_{\mu_\nu}$, which implies, for any $y \in B'_x$ and any multi-index J ,

$$(22) \quad |D^J f_x(y)| \leq \|f_x\|_{\mu_\nu} (\mu_\nu)^j j! M_j.$$

Combining (20), (21) and (22), we get the desired estimate (13) with suitable constants $A = C'_\nu$ and $B = \mu_\nu$ depending only on ν , hence only on K and λ .

We now prove the implication $(A') \Rightarrow (B')$. By Lemma 2.3, the assumption implies that $X \setminus X_\infty$ is dense in ∂X . The proof of the inclusion $\underline{m}_{X,M}^\infty \subset \mathcal{I}$ is a variant of the proof of Theorem 2.3 in [10]; we give some details for the reader's convenience. Let f be an element of $\underline{m}_{X,M}^\infty$. For any $x \in \Omega \setminus X$ and any multi-index $P \in \mathbb{N}^n$, the Leibniz formula yields

$$(23) \quad D^P(f/\varphi)(x) = \sum_{I+J=P} \frac{P!}{I!J!} D^I f(x) D^J(1/\varphi)(x).$$

Let K be a compact subset of Ω . For $x \in K \setminus X$, we combine the \mathcal{C}_M Łojasiewicz condition with Lemma 1.9 in order to obtain an estimate for all the terms $D^I f(x) D^J(1/\varphi)(x)$ that appear in (23). Lemma 1.9, together with (8), yields $|D^I f(x)| \leq c_1 c_2^i i! M_i (h_M(c_3 \text{dist}(x, X)))^2$ with $c_3 = \rho c_2$. Applying the \mathcal{C}_M Łojasiewicz condition with $\lambda = c_3$, we therefore get $|D^I f(x) D^J(1/\varphi)(x)| \leq c_2 C c_2^i \sigma^j i! j! M_i M_j h_M(c_3 \text{dist}(x, X))$. Since $i + j = p$, we have $i! j! \leq p!$, as well as $M_i M_j \leq M_p$ by (4). Gathering these estimates in (23), we obtain, for every multi-index P and every $x \in K \setminus X$,

$$(24) \quad |D^P(f/\varphi)(x)| \leq c_5 c_6^p p! M_p h_M(c_3 \text{dist}(x, X))$$

with $c_5 = c_2 C$ and $c_6 = c_2 + \sigma$. Using (24) and the Hesténès lemma, we see that the function g defined by $g(x) = f(x)/\varphi(x)$ for $x \in \Omega \setminus X$ and $g(x) = 0$ for $x \in X$, belongs to $\mathcal{C}_M(\Omega)$. Obviously, we have $f = \varphi g$, hence $f \in \mathcal{I}$.

Finally, we prove the implication $(B') \Rightarrow (C')$. Let f be an element of $\underline{m}_{X,M}^\infty$. By assumption, there is $g \in \mathcal{C}_M(\Omega)$ such that $f = \varphi g$. Let a be a point of $X \setminus X_\infty$. In the ring of formal power series, we have $0 = T_a f = (T_a \varphi)(T_a g)$ with $T_a \varphi \neq 0$, which implies $T_a g = 0$. Thus, g is flat on $X \setminus X_\infty$, hence on ∂X since it is assumed that $X \setminus X_\infty$ is dense in ∂X . Put $\tilde{g}(x) = g(x)$ for $x \in \Omega \setminus X$ and $\tilde{g}(x) = 0$ for $x \in X$. By the Hesténès lemma, it is then readily seen that \tilde{g} belongs to $\underline{m}_{X,M}^\infty$. Moreover, we have $f = \varphi \tilde{g}$, hence f belongs to $\mathcal{I} \underline{m}_{X,M}^\infty$, and the proof is complete. \square

Remark 2.5. We don't know whether the implication $(B') \Rightarrow (C')$ still holds without the additional assumption on $X \setminus X_\infty$ in (B') . This is true when X is a real-analytic submanifold of Ω : indeed, according to Theorem 4.2.4 of [13], we then have¹ $\underline{m}_{X,M}^\infty = \underline{m}_{X,M}^\infty \underline{m}_{X,M}^\infty$. Thus, in this case, the inclusion $\underline{m}_{X,M}^\infty \subset \mathcal{I}$ easily implies (C') .

Remark 2.6. Using the equivalence $(A') \Leftrightarrow (C')$, we see that if φ satisfies the \mathcal{C}_M Łojasiewicz condition and if h is an invertible element of the algebra $\mathcal{C}_M(\Omega)$, so that φ and $h\varphi$ generate the same ideal \mathcal{I} , then $h\varphi$ also satisfies the

¹The result in [13] is actually a local version of that statement, but it can be globalized, using partitions of unity.

\mathcal{C}_M Łojasiewicz condition. This can also be checked by a direct computation with the Leibniz formula.

3. ADDITIONAL PROPERTIES AND EXAMPLES

3.1. On the zero set. We have a Denjoy-Carleman counterpart of Proposition V.4.6 of [15].

Proposition 3.2. *Let φ be an element of $\mathcal{C}_M(\Omega)$ that satisfies the \mathcal{C}_M Łojasiewicz condition, and let X be its zero set. Then there is a \mathcal{C}_M -smooth submanifold Y of Ω such that $X = \overline{Y}$.*

Proof. We notice first that the conclusion of Lemma 2.3 only requires a weaker property than the \mathcal{C}_M Łojasiewicz condition: more precisely, the proof remains valid as soon as, for any compact subset K of Ω and any real $\lambda > 0$, one can find a constant $C > 0$ such that the inequality $|\varphi(x)| \geq Ch_M(\lambda \text{dist}(x, X))$ holds for any $x \in K$. It is then fairly easy to check that the proof by induction given in [15] for the usual Łojasiewicz inequality on \mathcal{C}^∞ functions remains valid in the \mathcal{C}_M case, up to minor modifications. \square

3.3. Connection with closedness. In this section, we show that the \mathcal{C}_M Łojasiewicz condition behaves as expected with respect to closedness properties of ideals.

Proposition 3.4. *Let φ be a non-zero element of $\mathcal{C}_M(\Omega)$ that generates a closed ideal in $\mathcal{C}_M(\Omega)$. Then φ satisfies the \mathcal{C}_M Łojasiewicz condition. Moreover, both properties are equivalent when the zeros of φ are isolated.*

Proof. We use the same notation as in the proof of the implication $(C') \Rightarrow (A')$ of Theorem 2.4. Put $\mathcal{I} = \varphi\mathcal{C}_M(\Omega)$ and assume that \mathcal{I} is closed in $\mathcal{C}_M(\Omega)$. Since the inclusion $\mathcal{D}_M(\Omega) \hookrightarrow \mathcal{C}_M(\Omega)$ is continuous, $\mathcal{I} \cap \mathcal{D}_M(\Omega)$ is closed in $\mathcal{D}_M(\Omega)$. Using cutoff functions, it is also easy to see that $\mathcal{I} \cap \mathcal{D}_M(\Omega) = \varphi\mathcal{D}_M(\Omega)$. It is then possible to duplicate the proof of the implication $(C') \Rightarrow (A')$ of Theorem 2.4, the only difference being that the map $f \mapsto \varphi f$ is now considered as a map from the (DFS)-space $\mathcal{D}_M(\Omega)$ onto its closed subspace $\varphi\mathcal{D}_M(\Omega)$.

The converse in the case of isolated zeros is based on a variant of the argument leading to Proposition 4.1 of [10] (which deals with a singleton). Assume that φ satisfies the \mathcal{C}_M Łojasiewicz condition and that its zero set X consists of isolated points, so that X is a countable subset $\{a_j : j \geq 1\}$ of Ω . Put $\mathcal{I} = \varphi\mathcal{C}_M(\Omega)$ and let f be an element of the closure $\overline{\mathcal{I}}$. By the \mathcal{C}_M version of Whitney's spectral theorem [4], for every $j \geq 1$ there is a function g_j of $\mathcal{C}_M(\Omega)$ such that $f - \varphi g_j$ is flat at a_j . Let $(\chi_j)_{j \geq 1}$ be a sequence of compactly supported elements of $\mathcal{C}_M(\Omega)$ such that $\chi_j = 1$ in a neighborhood of a_j and $\text{supp } \chi_j \cap \text{supp } \chi_k = \emptyset$ for $k \neq j$. Then the (locally finite) series $g = \sum_{j \geq 1} \chi_j g_j$ defines an element of $\mathcal{C}_M(\Omega)$ and we have $f - \varphi g \in \underline{m}_{X,M}^\infty$. Since (B') holds, this yields $f \in \mathcal{I}$, hence the result. \square

Example 3.5. According to Proposition 3.4 and the results in [10, 12], examples of functions φ which satisfy the \mathcal{C}_M Łojasiewicz condition will include any

homogeneous polynomial with an isolated real critical point at 0, as well as real analytic functions whose germs of complex zeros intersect \mathbb{R}^n at isolated points with Łojasiewicz exponent 1 for the regular separation property. On the other hand, some analytic functions do not satisfy the \mathcal{C}_M Łojasiewicz condition: for instance, given an integer $k \geq 2$, the polynomial $\psi(x) = x_1^2 + x_2^{2k}$ does not satisfy the \mathcal{C}_M Łojasiewicz condition in \mathbb{R}^2 , as can be seen from the results in [10] (property (B') fails).

We now give an example showing that the converse to Proposition 3.4 is false without the assumption of isolated zeros. In particular, the \mathcal{C}_M Łojasiewicz condition does not imply closedness in general.

Example 3.6. We put $n = 2$, $\Omega = \mathbb{R}^2$, and $\varphi(x) = x_1\psi(x)$ where ψ is the polynomial mentioned in Example 3.5. We then have $X = \{x \in \mathbb{R}^2 : x_1 = 0\}$ and $\text{dist}(x, X) = |x_1|$. Let x be a point in $\mathbb{R}^2 \setminus X$. For any $v = (v_1, v_2) \in \mathbb{C}^2$, we have

$$|\psi(x+v) - \psi(x)| \leq 2|x_1||v_1| + |v_1|^2 + \sum_{p=1}^{2k} \binom{2k}{p} |x_2|^{2k-p} |v_2|^p.$$

We also have the obvious inequalities $|x_1| \leq (\psi(x))^{1/2}$ and $|x_2| \leq (\psi(x))^{1/2k}$. Thus, if we assume $|v_1| \leq \delta(\psi(x))^{1/2}$ and $|v_2| \leq \delta(\psi(x))^{1/2k}$ for some real number δ with $0 < \delta < 1$, we get

$$|\psi(x+v) - \psi(x)| \leq \left(2\delta + \delta^2 + \sum_{p=1}^{2k} \binom{2k}{p} \delta^p\right) \psi(x) \leq (2^{2k} + 2)\delta\psi(x).$$

Setting $\delta = (2^{2k+1} + 4)^{-1}$, we obtain $|\psi(\zeta)| \geq \frac{1}{2}\psi(x)$ for every point ζ in the bidisc $\{\zeta \in \mathbb{C}^2 : |\zeta_1 - x_1| \leq \delta(\psi(x))^{1/2}, |\zeta_2 - x_2| \leq \delta(\psi(x))^{1/2k}\}$. The Cauchy formula then yields, for every $(i, j) \in \mathbb{N}^2$,

$$\left| \frac{\partial^{i+j}}{\partial x_1^i \partial x_2^j} \left(\frac{1}{\psi(x)} \right) \right| \leq 2\delta^{-(i+j)} i!j! (\psi(x))^{-(\frac{i}{2} + \frac{j}{2k} + 1)},$$

which easily implies

$$(25) \quad \left| \frac{\partial^{i+j}}{\partial x_1^i \partial x_2^j} \left(\frac{1}{\psi(x)} \right) \right| \leq 2\delta^{-(i+j)} i!j! |x_1|^{-(i+j+2)}$$

provided we assume $|x_1| < 1$. Using (25), the definition of φ , and the Leibniz formula, we then get

$$\left| \frac{\partial^{i+j}}{\partial x_1^i \partial x_2^j} \left(\frac{1}{\varphi(x)} \right) \right| \leq B^{i+j+1} i!j! |x_1|^{-(i+j+2)}$$

for some suitable constant $B > 0$. Given $\lambda > 0$, we then write $|x_1|^{-(i+j+2)} = \frac{\lambda^{i+j+2} M_{i+j+2}}{(\lambda|x_1|)^{i+j+2} M_{i+j+2}}$. We have $(\lambda|x_1|)^{i+j+2} M_{i+j+2} \geq h_M(\lambda|x_1|) = h_M(\lambda \text{dist}(x, X))$ by definition of h_M , whereas (5) yields $M_{i+j+2} \leq A^{i+j+2} M_2 M_{i+j}$. Thus, we

obtain $|x_1|^{-(i+j+2)} \leq (A\lambda)^{i+j+2}(h_M(\lambda \operatorname{dist}(x, X)))^{-1}$. We eventually get

$$\left| \frac{\partial^{i+j}}{\partial x_1^i \partial x_2^j} \left(\frac{1}{\varphi(x)} \right) \right| \leq \frac{C\sigma^{i+j}(i+j)!M_{i+j}}{h_M(\lambda \operatorname{dist}(x, X))}$$

with $C = A^2B\lambda^2$ and $\sigma = AB\lambda$. Thus, we have established the desired estimate for $|x_1| = \operatorname{dist}(x, X) < 1$, which suffices to claim that φ satisfies the \mathcal{C}_M Łojasiewicz condition (see Remark 2.2). However, the ideal $\mathcal{I} = \varphi\mathcal{C}_M(\mathbb{R}^2)$ is not closed for $k \geq 2$. Indeed, in this case, it has been shown in [10] that the ideal $\mathcal{J} = \psi\mathcal{C}_M(\mathbb{R}^2)$ is not closed. Since \mathcal{J} is the preimage of \mathcal{I} under the continuous mapping $\Pi : \mathcal{C}_M(\mathbb{R}^2) \rightarrow \mathcal{C}_M(\mathbb{R}^2)$ defined by $\Pi(f)(x) = x_1f(x)$, we see that \mathcal{I} is not closed either.

We conclude with a natural question.

Problem. Is it possible to extend the above results to the general case of finitely generated ideals? A first idea is to mimic the definition of Łojasiewicz ideals in the \mathcal{C}^∞ case, and say that a finitely generated ideal of $\mathcal{C}_M(\Omega)$ is Łojasiewicz if it contains an element φ which satisfies the \mathcal{C}_M Łojasiewicz condition. However, this definition doesn't seem to allow an immediate extension of the crucial implication $(C') \Rightarrow (A')$, whose proof is quite different from the \mathcal{C}^∞ case and doesn't seem easily adaptable to the case of several generators.

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